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A New Poisson Inverted Exponential Distribution: Model, Properties and Application

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ABSTRACT
A new Poisson Inverted Exponential distribution is developed from the Poisson family of distribution, which has two parameters. The characteristic of the intended model is unimodal, positive skewed and platykurtic, while the characteristic of the hazard function is the inverted bathtub and the decreasing order. Explicit expression of quantile function, moments (including incomplete and conditional moments), moment generating function, residual life function, R`enyi and q-entropies, probability weighted moment and order statistics of the intended model. The value of unknown parameters is estimated by the maximum likelihood estimate with the confidence interval. Similarly, purposed model compared with well-known other five distributions through different criteria like as goodness of fit, P-P plot, Q-Q plots and K-S test. Likewise, we fitted the PDF and CDF of purposed model with other models, it is clear that intended model is great flexibility and satisfactory fit than those models. Therefore purposed model is more useful in real data and life time data analysis and modelling.

KEYWORDS: Inverted Exponential-Poisson, maximum likelihood estimation, order statistics

INTRODUCTION

Over the last decade, several probability distributions have been commonly used in real data modelling and forecasts in applied science, engineering, actuarial science, economics, telecommunications, life testing, and others many areas (Abouelmagd et al., 2017; Garrido et al., 2016; Soliman, et al. 2017). In literature, some familiar distribution has been derived, which are used in real data analysis in different areas are: Generalized Exponential-Poisson by Barreto-Souza & Cribari-Neto (2009), Gemotric exponential Poission G by Nadarajah et al. (2013), Exponentiated exponential Poisson G family by Ristić & Nadarajah (2014), Kumaraswamy Poisson-G Family by Ramos et al. (2015), Exponentiated generalized-G Poisson family by Aryal et al. (2017), Poission exponential –G family by Rayad et al. (2020), and others.

In the last few decades, the new probability distribution has been derived continuously. The new distribution having more parameter and flexibility than existing one; thus, these distributions are more robust and consistent. Therefore, new distributions are better fitted with complex data, which is the prime objective of introducing the new probability distribution. For this reason, a new two parameters Poisson Inverted Exponential (PIE) distribution has been derived.

METHODS AND MATERIALS

Poisson Inverted Exponential distribution

The exponential distribution has been extended in numerous ways to get new probabilistic models for life testing problems. Let, Y follows an exponential distribution, the distribution, $X = \frac{1}{Y}$ would be an inverted exponential distribution. It is used as a prospective life distribution analysis. Therefore, Cumulative Density Function (CDF) of Inverted Exponential (IE) distribution is given by,

$$G(x) = e^{-\frac{\beta}{x}}, x > 0, \beta > 0 \tag{2.1}$$

In literature, not only probability distribution, but also family of distribution has been derived. Chakraborty et al. (2020), defined Poisson-G family of distributions which CDF is given as

$$F(x) = \frac{1 - e^{-\lambda G(x)}}{1 - e^{-\lambda}}, x > 0, \lambda > 0 \tag{2.2}$$

The CDF of IE having parameter β in the equation (2.1) which is used in the equation (2.2), then CDF of new Poisson Inverted Exponential (PIE) distribution having 2 parameters becomes;

$$F(x) = \frac{1}{\left(1 - e^{-\lambda}\right)} e^{-\lambda\left(e^{-\frac{\beta}{x}}\right)};\ x > 0, \beta > 0, \lambda > 0 \tag{2.3}$$

The PDF of purposed model becomes,

$$f(x;\beta,\lambda) = \frac{\beta\lambda}{x^2\left(1 - e^{-\lambda}\right)} e^{-\frac{\beta}{x}} e^{-\lambda e^{-\frac{\beta}{x}}}; x > 0, \beta > 0, \lambda > 0 \tag{2.4}$$

The survival function is

$$\text{R(x)} = \frac{e^{-\lambda e^{-\frac{\beta}{x}}} - e^{-\lambda}}{1 - e^{-\lambda}}, x > 0, \beta > 0, \lambda > 0 \tag{2.5}$$

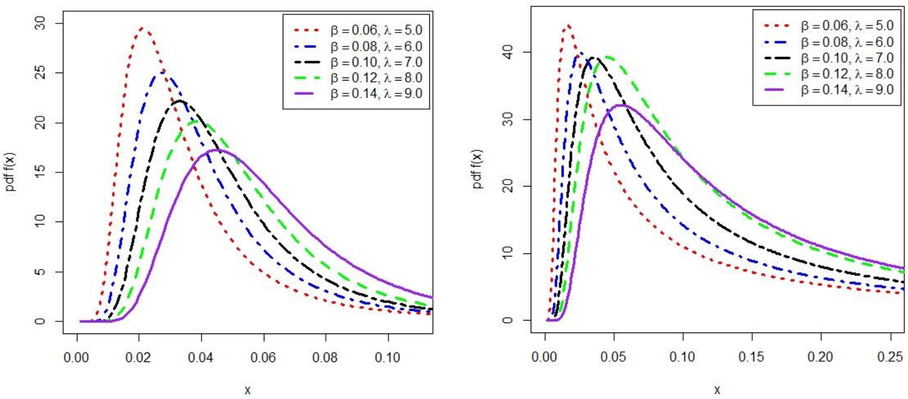
The hazard rate function is

$$\text{h(x)} = \frac{\beta\lambda\ e^{-\frac{\beta}{x}} e^{-\lambda e^{-\frac{\beta}{x}}}}{x^2\left[e^{-\lambda e^{-\frac{\beta}{x}}} - e^{-\lambda}\right]}; x > 0, \beta > 0, \lambda > 0 \tag{2.6}$$

It is noted that this model is quite flexible for modeling either real data or life-time data, which may be positive and skewed in nature (Fig 1, left panel). The hazard rate function of this model is inverted bathtub and decreasing order, which follow the good statistical behavior of modeling in real data analysis (Fig 1, right panel).

Figure 1

Plot of Probability Density Function (left panel) and Hazard Rate Function



Note: Plot of Probability density function (left panel) and hazard rate function (right panel) of some parameters value of (β, λ)

Statistical Properties

In this section major properties of PIE distribution have been derived.

Useful expansions

For n, a positive real non integer and $|z| < 1$, we have the generalized binomial series,

$$(1-z)^n = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} Z^j \quad , \text{for } n > 0 \tag{3.1}$$

The power series of exponential function is

$$e^{-ax} = \sum_{i=0}^{\infty} \frac{(-1)^i (a\,x)^i}{i!} \tag{3.2}$$

Using the exponential power series (3.2) in equation (2.4), the PDF of new PIE distribution becomes,

$$f(x) = \sum_{i=0}^{\infty} \Psi_i \frac{e^{-\frac{\beta}{x}(1+i)}}{x^2} \tag{3.3}$$

$$\text{Where, } \Psi_i = \frac{(-1)^i \beta \lambda^{i+1}}{i!(1-e^{-\lambda})}$$

Similarly, $\left[f(x) \right]^{\delta} = \sum_{k=0}^{\infty} \frac{\Psi_k^* e^{-\frac{\beta}{x}(\delta+k)}}{x^{2\delta}}$ (3.4)

$$\text{Where, } \Psi_k^* = \frac{(-1)^k \delta^k \beta^{\delta} \lambda^{\delta+k}}{k!\left(1-\mathrm{e}^{-\lambda}\right)^{\delta}}$$

Likewise, binomial expansion (3.1) and exponential power series expansion (3.2) is used in the expansion of $\left[F\left(x\right)\right]^{s}$, then it becomes,

$$\left[F\left(x\right)\right]^{s} = \sum_{k=0}^{\infty} \omega_{jk} e^{-\frac{\beta \mathrm{k}}{x}} \quad (3.5)$$

$$\text{Where, } \omega_{jk} = \sum_{j=0}^{\infty} \frac{(-1)^{j+k} (\lambda j)^k}{\mathrm{k}!\left(1-e^{-\lambda}\right)^{s}} \binom{s}{j}$$

Quantile function

Quantile functions are used for theoretical aspects of probability. The quantile function is defined of any distribution is $Q(u) = F^{-1}(u)$. Therefore, the corresponding quantile function for the purposed model is,

$$Q(u) = \frac{-\beta}{\ell n\left[\frac{-\ell n\{1-u(1-e^{-\lambda})\}}{\lambda}\right]} \quad (3.6)$$

Where u ~ U (0, 1) distribution.

Now, equation (3.6) is used to generate 100 random samples and it describe the characteristic of distribution, like mean, median, mode, skewness, and kurtosis. It is observed that the distribution is unimodal, positively skewed, and platykurtic in nature. As a result, any data set follows such a feature, the intended model is particularly suitable in modeling of real and life time data (Table 1, Fig 1).

Table 1

The mean, Median, Skewness and Kurtosis for Different Values of the Parameters

Parameters		Mean	Median	Mode	Skewness	Kurtosis
β	λ					
0.01	5.0	0.437214	0.361934	0.957946	0.364908	1.631523
0.02		0.459600	0.384203	1.02188	0.332095	1.562768
0.03	4.0	0.481015	0.396426	1.079046	0.296169	1.491212
0.04	3.5	0.500193	0.408688	1.123853	0.257066	1.422893
0.05	3.0	0.516207	0.436164	1.152273	0.218044	1.366162
0.06	2.5	0.565164	0.551858	1.934498	0.183327	1.326639
0.07	2.0	0.536839	0.460977	1.152919	0.156606	1.306093
0.08	1.5	0.541283	0.491424	1.125854	0.140277	1.303092
0.09	1.0	0.542079	0.517447	1.083159	0.135295	1.314182
0.10	0.5	0.539607	0.490153	1.027781	0.141264	1.335081

Moments

Since, $X \sim PIE(\beta,\lambda)$, the r[th] raw moment of random variable is given by

$$E(X^r)=\mu_r'=\int_o^\infty x^r f(x)dx=\sum_{i=0}^{\infty}\Psi_i\int_0^\infty x^{r-2}e^{-\frac{\beta}{x}(1+i)}\,dx \qquad (3.7)$$

Where, PDF is used from the equation (3.3) then moments of PIE distribution is

$$\mu_r'=\sum_{i=0}^{\infty}\frac{\Psi_i\Gamma(1-r)}{\left[\beta(1+i)\right]^{1-r}}$$

Similarly, lower incomplete moments, say, $\varphi_s(t)$ is given by

$$\varphi_s(t)=\int_0^t x^s f(x)dx=\sum_{i=0}^{\infty}\Psi_i\int_0^t x^{s-2}e^{-\frac{\beta}{x}(1+i)}dx \qquad (3.8)$$

Using $\Gamma(s,\mathrm{x})=\int_x^\infty t^{s-1}e^{-t}dt$ in equation (3.8), after substitute $z=\frac{1}{x}$ then $\varphi_s(t)$ becomes;

$$\varphi_s(t)=\sum_{i=0}^{\infty}\Psi_i\frac{\Gamma\left(1-s,\ {}^{1}\!/_{t}\left[\beta(1+i)\right]\right)}{\left[\beta(1+i)\right]^{(1-s)}}$$

Again, conditional moment is $\tau_s(t)=\int_t^\infty x^s f(x)dx=\sum_{i=0}^{\infty}\Psi_i\int_t^\infty x^{s-2}e^{-\frac{\beta}{x}(1+i)}dx$ (3.9)

Using $\gamma(s,\mathrm{x})=\int_0^x t^{s-1}e^{-t}dt$ in equation (3.9), after substitute $z=\frac{1}{x}$ then $\tau_s(t)$ becomes

$$\tau_s(t)=\sum_{i=0}^{\infty}\Psi_i\frac{\gamma\left(1-s,\ {}^{1}\!/_{t}\left[\beta(1+i)\right]\right)}{\left[\beta(1+i)\right]^{(1-s)}}$$

Moment Generating Function (MGF)

The moment generating function is $M_x(t)=E(e^{tx})=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(\mathrm{x}^r)$ (3.10)

After using the finding of (3.7) in equation (3.10), the MGF is

$$M_x(t)=\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^r\Psi_i\Gamma(1-r)}{r!\left[\beta(1+i)\right]^{1-r}}$$

Residual life function

The n[th] moment of the residual life of X is given by

$$m_n(t) = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x) dx \qquad (3.11)$$

Apply the binomial expansion of $(x-t)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{n-r} t^r$ in to equation (3.11),

we get $m_n(t) = \frac{1}{R(t)} \sum_{i=0}^{\infty} \sum_{r=0}^{n} (-t)^r \Psi_i \binom{n}{r} \int_t^{\infty} x^{n-r-2} e^{-\frac{\beta}{x}(1+i)} dx$ (3.12)

Using $\gamma(s, \mathrm{x}) = \int_0^x t^{s-1} e^{-t} dt$ after substitute, $z = \frac{1}{x}$ then, nth moment of the residual life function is

$$m_n(t) = \frac{1}{R(t)} \sum_{i=0}^{\infty} \sum_{r=0}^{n} (-t)^r \binom{n}{r} \Psi_i \frac{\gamma\left(1-n+r, \frac{1}{t}\left[\beta(1+i)\right]\right)}{\left[\beta(1+i)\right]^{1-n+r}}$$

R\`enyi and *q*-entropies

The entropy of a random variable is a measure of variation of uncertainty or randomness of a system. The theory of entropy has been used in many fields such as physics, engineering, economics, and other subjects (Song, 2001). It is used in statistics for testing hypothesis in parametric models and lifetime distribution (Al-saiary et al., 2019). The R\`enyi entropy is defined as

$$I_\delta(\mathrm{X}) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} \left[f(x) \right]^\delta dx, \quad \delta > o \;\&\; \delta \neq 1 \qquad (3.13)$$

By applying the relation (3.4) in equation (3.13) and integration of (3.13), then finding of R\`enyi entropy is obtained as,

$$I_\delta(\mathrm{X}) = \frac{1}{1-\delta} \log \left[\sum_{k=0}^{\infty} \Psi_k^* \frac{\Gamma(2\delta-1)}{\left[\beta(\delta+k)\right]^{(2\delta-1)}} \right]$$

Similarly, the q-entropy is defined by

$$\mathrm{H}_q(\mathrm{X}) = \frac{1}{1-q} \log \left(1 - \int_{-\infty}^{\infty} \left[f(x) \right]^q dx \right), \quad \mathrm{q} > o \;\&\; \mathrm{q} \neq 1 \qquad (3.14)$$

Therefore, q- entropy of PIE distribution is obtained by substituting the result of (3.13) in to (3.14) only when replacing δ by q, we get

$$H_q(\mathrm{X}) = \frac{1}{1-q} \log \left[1 - \sum_{k=0}^{\infty} \Psi_k^* \frac{\Gamma(2q-1)}{\left[\beta(\mathrm{q}+k)\right]^{(2q-1)}} \right]$$

The Probability Weighted Moments (PWM)

The probability weighted moments can be obtained from the following relation

$$\tau_{r,s} = E(X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x) F(x)^s dx \qquad (3.15)$$

By substituting equations (3.3) and (3.5) in equation (3.15) and integrating the equation (3.15), then finding of PWM is;

$$\tau_{r,s} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \Psi_i \omega_{jk} \frac{\Gamma(1-r)}{[\beta(1+i+k)]^{(1-r)}}$$

Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let, $\varkappa_{(1)} < \varkappa_{(2)} < \ldots < \varkappa_{(n)}$ denotes the order statistic of a random sample $X_1, X_2 \ldots X_n$ from the PIE distribution with CDF F(x) and PDF f(x). According to H. A. David (as cited in Al-saiary et al., 2019), the PDF of $X_{(r)}$ can be written as

$$f_r(x_{(r)}) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \qquad (3.16)$$

By using PDF in equation (3.3) and expansion of (2.3) by using exponential power series expansion, then equation (3.16) becomes,

$$f_r(x_{(r)}) = M\sum_{i=0}^{\infty}\Psi_i \frac{e^{-\frac{\beta}{x_{(r)}}(1+i)}}{x_{(r)}^2}\left[\frac{1}{1-e^{-\lambda}}\left\{1-\sum_{i=0}^{\infty} n_i\, e^{\left(-\frac{\beta}{x_{(r)}}\right)^i}\right\}\right]^{r-1}\left[\frac{1}{1-e^{-\lambda}}\left\{\sum_{i=0}^{\infty} n_i\, e^{\left(-\frac{\beta}{x_{(r)}}\right)^i} - e^{\lambda}\right\}\right]^{n-r} ; x_{(r)} > 0 \qquad (3.17)$$

Where, $M = \frac{n!}{(r-1)!(n-r)!}$ and $n_i = \frac{\lambda^i}{i!}$. Therefore, The PDF of largest order statistics $X_{(n)}$ is,

$$f_n(x_{(n)}) = n\sum_{i=0}^{\infty}\Psi_i \frac{e^{-\frac{\beta}{x_{(n)}}(1+i)}}{x_{(n)}^2}\left[\frac{1}{1-e^{-\lambda}}\left\{1-\sum_{i=0}^{\infty} n_i\, e^{\left(-\frac{\beta}{x_{(1)}}\right)^i}\right\}\right]^{n-1} ; x_{(n)} > 0$$

The PDF of smallest order statistics $X_{(1)}$ is;

$$f_1(x_{(1)}) = n\sum_{i=0}^{\infty}\Psi_i \frac{e^{-\frac{\beta}{x_{(1)}}(1+i)}}{x_{(1)}^2}\left[\frac{1}{1-e^{-\lambda}}\left\{\sum_{i=0}^{\infty} n_i\, e^{\left(-\frac{\beta}{x_{(1)}}\right)^i} - e^{\lambda}\right\}\right]^{n-1} ; x_{(1)} > 0$$

Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) of the unknown's parameters of the distribution based on $\underset{\sim}{x} = (x_1, \ldots, x_n)$ observed sample with the set of parameters $\ell(\beta, \lambda \mid \underset{\sim}{x})$. The log likelihood function of the parameter $\ell(\beta, \lambda)$ is given by

$$\ell n(x) = n\ell n(\beta\lambda) - n\ell n\left(1-e^{-\lambda}\right) - \beta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) - 2\sum_{i=1}^{n}\ell n(x_i) - \lambda\sum_{i=1}^{n}\left(e^{-\frac{\beta}{x_i}}\right) \quad (4.1)$$

Maximum likelihood estimators of the parameters have obtained by partial differentiating w.r.t.to parameters and equating to zero, we have

$$\frac{\partial \ell n(\ell)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) - \lambda\sum_{i=1}^{n}\left(\frac{e^{-\frac{\beta}{x_i}}}{x_i}\right) = 0 \quad (4.2)$$

$$\frac{\partial \ell n(\ell)}{\partial \lambda} = \frac{n}{\lambda} - \frac{n}{e^{\lambda}-1} - \sum_{i=1}^{n}\left[e^{-\left(\frac{\beta}{x_i}\right)}\right] = 0 \quad (4.3)$$

The unknown parameters β and λ are estimated by solving non-linear equation (4.2) and (4.3). Clearly, it is difficult to solve them analytically; therefore, applying Newton-Raphson's iterative technique by using *optim* () function in R software (Braun et al., 2016), R core team (2019).

Hence, from the asymptotic normality of MLE's, approximate $100\,(1-\gamma)$ % confidence interval for β and λ can be constructed as

$$\hat{\beta} \pm z_{\gamma/2}\sqrt{\text{var}(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\gamma/2}\sqrt{\text{var}(\hat{\lambda})}$$

Where, $Z_{\gamma/2}$ is the upper percentile of standard normal variate.

RESULTS AND DISCUSSION

In this section, the analysis of one real data set for illustration of purposed model is presented. In this paper, the author considers a data set purposed by Hinkley (1977) which are given as:
0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Parameter Estimation

The value of parameters are estimated by maximizing the log-likelihood function (4.1) directly by using *optim*() function in R software. Also, author estimate $100\,(1-\gamma)$ % confidence interval of purposed model (Table 2).

Table 2

Estimated Value, SE and 95% Confidence Interval of Parameters

Parameters	MLE	SE	95% CI
$\hat{\beta}$	2.507471	0.4537	(1.61821,3.39672)
$\hat{\lambda}$	4.605174	1.5881	(1.49249,7.71785)

Model Comparisons

We have considered five alternative models named Flexible Weibull (FW) by Bebbington et al. (2007), Expononented Inverted Weibull (EIW) by Flaih et al. (2012), Generalized Inverted Exponential (GIE) by Krishna and Kumar (2013), Weighted

Inverted Exponential (WIE) by Hussein, (2013) and Type II half logistic exponential by Elgarhy et al. (2019) compare with the purposed model. These models are compared with our model by different goodness of fit criteria's like as (i) Akaike's information criterion (AIC), (ii) Bayesian information criterion (BIC), (iii) Corrected Akaike's information criterion (CAIC) and (iv) Hannan-Quinn Information Criterion (HQIC) (Table 3).

Table 3

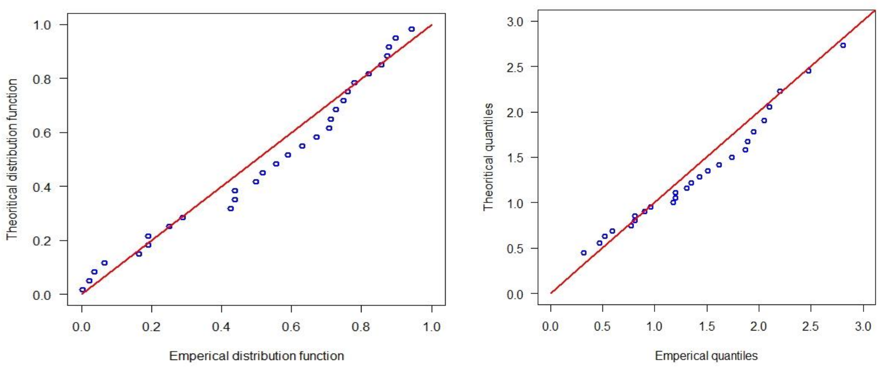
Comparison of Purposed Model with Known Distribution by Goodness of fit Criteria

Models	AIC	BIC	CAIC	HQIC
PIE	65.41938	68.22177	65.86328	66.31589
WIE	86.10524	88.90763	86.54968	87.00175
EWI	87.83402	90.63642	88.27847	88.73053
FW	80.13606	82.93845	80.58050	81.03257
GIF	83.31921	86.12160	83.76365	84.21572

The value of AIC, BIC, CIAC, and HQIC is the least of purposed model. Hence, purposed model is a better fit for positively skewed data. Similarly, goodness of fit is also verified by P-P plot and Q-Q plot. Likewise, the KS test was 0.12733 (p-value=0.378) of purposed model, indicating that it is good fit, (Kumar and Ligges, 2011) (Figure 2).

Figure 2

P-P plot (left panel) and Q-Q plot (right panel) of Purposed Model

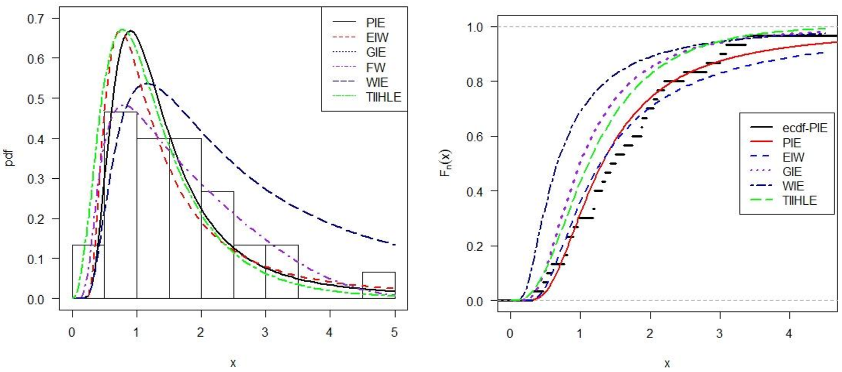


Note: P-P plot (left panel) and Q-Q plot (right panel) of purposed model, used by estimated MLE

Likewise, author compare the purposed model with other known distribution by fitting PDF and CDF from estimated MLEs. In Figure 3, it is clear that the intended model is better fitted than other known distributions. Hence, purposed model is an alternative, greater flexible model for real data and life time data modeling.

Figure 3

Estimated Fitted Densities, Estimated CDFs and Empirical CDF



Note: Estimated Fitted Densities (left panel), Estimated CDFs and Empirical CDF (right panel)

CONCLUSION

In this study, a new Poisson Inverted Exponential distribution with two parameters has been discussed. Some of the important properties of the distribution name, quintile, moments, moment generating function, residual life function, R`enyi and *q* entropy, probability weighted moments and order statistic are investigated of intended model. The values of parameters are estimated from maximum likelihood methods with confidence interval. From the data analysis, it is observed that PIE distribution is a better than others some well-known distribution. Hence, purposed model is a satisfactory model in both aspects i.e. the theoretical and applied in real data and life time data modeling.

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